

Fake proofs for identities involving products of Eisenstein series

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ABSTRACT. In the workshop of the July 2016 Building Bridges 3 conference in Sarajevo, I presented the results from a joint article with W. Raji (Mathematische Annalen, to appear). That article gave a proof of various linear relations between products of two Eisenstein series on $\Gamma(N)$, including an interesting identity related to the action of a Hecke operator on such a product. The real proofs involve some care to deal with issues of convergence. In this note we give “fake” proofs for these identities, ignoring the convergence issues; some of these fake proofs appeared in the workshop lecture as an amusing side note before I sketched the real proofs. Something in these fake proofs is quite suggestive, even though the proofs themselves are clearly invalid (and even produce wrong results). It would be interesting to understand what exactly is going on here.

1. Introduction

The basic object of study in this note is the Eisenstein series of weight $\ell \geq 1$ on $\Gamma(N)$, with parameter $\lambda \in N^{-1}\mathbf{Z}^2/\mathbf{Z}^2 \subset \mathbf{Q}^2/\mathbf{Z}^2$:

$$(1.1) \quad E_{\ell,\lambda}(z) = \sum_{\substack{(a,b) \equiv \lambda \pmod{\mathbf{Z}^2} \\ (a,b) \neq (0,0)}} (az + b)^{-\ell}.$$

The above is not quite right if $\ell \in \{1, 2\}$, as the above series does not converge. In that case, one evaluates the sum, following Hecke, by replacing $(az + b)^{-\ell}$ by $(az + b)^{-\ell}|az + b|^{-s}$ for a complex parameter s , and then setting $s = 0$ after one has analytically continued the resulting sum in s . In the convergent case, when $\ell \geq 3$, the Eisenstein series is a holomorphic function of z , and this holomorphy fortuitously still holds for $\ell = 1$; however, for $\ell = 2$, Hecke’s summation procedure yields that $E_{2,\lambda}(z)$ is the sum of a nonholomorphic expression $-\pi/\text{Im } z$ (which is the same for all λ) with a holomorphic function (which of course depends on λ).

The product $E_{\ell,\lambda}E_{m,\mu}$ of two of these Eisenstein series is then a form of weight $\ell + m$ on $\Gamma(N)$. The articles [BG01, BG03] prove a number of linear relations between such products — more precisely, they show that certain linear combinations of such products belong to the space $Eis_{\ell+m}$ of Eisenstein series of

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weight $\ell + m$. These linear relations modulo $Eis_{\ell+m}$ have a structure that is reminiscent of the Manin relations between periods of cusp forms; this was further codified in [Pa06]. For example, in weight 2, we have the following relations:

$$(1.2) \quad \begin{aligned} E_{1,\lambda}E_{1,\mu} + E_{1,\mu}E_{1,-\lambda} &= 0, \\ E_{1,\lambda}E_{1,\mu} + E_{1,\mu}E_{1,-\lambda-\mu} + E_{1,-\lambda-\mu}E_{1,\lambda} &\equiv 0 \pmod{Eis_2}. \end{aligned}$$

The first identity, which is very simple, is analogous to the two-term Manin relation involving $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, while the second, more interesting, identity, is analogous to the three-term Manin relation involving $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$. The explanation for this parallelism can be found in [KMR17], where we show that the Petersson inner product of any cusp form $f \in \mathcal{S}_2$ with $E_{1,\lambda}E_{1,\mu} + E_{1,\mu}E_{1,-\lambda-\mu} + E_{1,-\lambda-\mu}E_{1,\lambda}$ expands to a combination of periods of f and its transforms $f|_2 M$ for certain $M \in GL_2^+(\mathbf{Q})$, and this combination of periods vanishes precisely by the Manin relations. The proof in that article, which works for a similar identity in arbitrary weight, involves carrying throughout the parameter s in the Eisenstein series $E(z, s)$, and controlling the analysis fairly carefully. Our goal in this note is to provide fake proofs of that and other results from [KMR17], relying on intriguing identities between rational functions, but with no attention paid to convergence. It would be interesting to find the connection between these intriguing identities and the structure of the Manin relations, and to see what parts of the fake proofs can be salvaged.

FAKE PROOF OF THE SECOND IDENTITY IN (1.2). Let us write $\nu = -\lambda - \mu$; hence we can assume given a triple (λ, μ, ν) for which $\lambda + \mu + \nu = (0, 0)$. Similarly, consider the set $T = T_{(\lambda, \mu, \nu)}$ of all triples $((a, b), (c, d), (e, f)) \in (\mathbf{Q}^2)^3$ with

$$(1.3) \quad \begin{aligned} (a, b) &\equiv \lambda \pmod{\mathbf{Z}^2}, \\ (c, d) &\equiv \mu \pmod{\mathbf{Z}^2}, \\ (e, f) &\equiv \nu \pmod{\mathbf{Z}^2}, \\ (a, b) + (c, d) + (e, f) &= (0, 0), \end{aligned}$$

None of (a, b) , (c, d) , or (e, f) equals $(0, 0)$.

(The set T is nonempty precisely because $\lambda + \mu + \nu = (0, 0)$, and the last condition of nonvanishing only matters if one of λ, μ, ν is zero in $\mathbf{Q}^2/\mathbf{Z}^2$.) Then, ignoring all issues of convergence, we formally have

$$(1.4) \quad E_{1,\lambda}E_{1,\mu} \equiv \sum_{((a,b),(c,d),(e,f)) \in T} \frac{1}{az+b} \cdot \frac{1}{cz+d} \pmod{Eis_2}.$$

The reason is that once one chooses (a, b) and (c, d) arbitrarily in the congruence classes of λ and μ , respectively, the pair $(e, f) = -(a, b) - (c, d)$ is uniquely determined. In the event that $(e, f) = (0, 0)$, which anyhow only occurs when $\lambda = -\mu$, the pairs that we omit are those with $(c, d) = (-a, -b)$, which corresponds to being off by $-E_{2,\lambda}$, which (ignoring its nonholomorphy) “is” an element of Eis_2 .

We obtain similar expressions (modulo Eis_2) for $E_{1,\mu}E_{1,\nu}$ and for $E_{1,\nu}E_{1,\lambda}$. Thus, working modulo Eis_2 , we obtain

$$(1.5) \quad \begin{aligned} &E_{1,\lambda}E_{1,\mu} + E_{1,\mu}E_{1,\nu} + E_{1,\nu}E_{1,\lambda} \\ &\equiv \sum_{((a,b),(c,d),(e,f)) \in T} \left[\frac{1}{(az+b)(cz+d)} + \frac{1}{(cz+d)(ez+f)} + \frac{1}{(ez+f)(az+b)} \right], \end{aligned}$$

and this last sum vanishes, thanks to the identity

$$(1.6) \quad p + q + r = 0 \implies \frac{1}{pq} + \frac{1}{qr} + \frac{1}{rp} = 0,$$

with $p = az + b$, $q = cz + d$, and $r = ez + f$. \square

It is not apparent to me how to salvage the above fake proof, for instance by summing over elements of T in a particular order so as to obtain convergence, in the style of Eisenstein [Wei99]. Indeed, when λ, μ, ν are all nonzero, then our fake proof would imply that $E_{1,\lambda}E_{1,\mu} + E_{1,\mu}E_{1,\nu} + E_{1,\nu}E_{1,\lambda}$ is actually zero, which is not the case; its expression as an explicit weight 2 Eisenstein series is known.

2. The identity in higher weight

We now turn to the case of general weight $k \geq 2$. The analog of the two-term Manin relation is again simple, and can be found in equation (2.23) of [KMR17]. The interesting three-term relation in higher weight amounts to the following.

PROPOSITION 2.1 (Theorem 2.8 of [KMR17]). *Let λ, μ, ν satisfy $\lambda + \mu + \nu = 0$, as before. Let $\alpha, \beta, \gamma \in \mathbf{C}$ satisfy $\alpha + \beta + \gamma = 0$ (these can be thought of as formal variables). Let $k \geq 2$. Then the following expression is orthogonal to all cusp forms $f \in \mathcal{S}_k$:*

$$(2.1) \quad \sum_{\substack{\ell+m=k \\ \ell, m \geq 1}} \alpha^{\ell-1} \beta^{m-1} E_{\ell,\lambda} E_{m,\mu} + \sum_{\substack{\ell+m=k \\ \ell, m \geq 1}} \beta^{\ell-1} \gamma^{m-1} E_{\ell,\mu} E_{m,\nu} + \sum_{\substack{\ell+m=k \\ \ell, m \geq 1}} \gamma^{\ell-1} \alpha^{m-1} E_{\ell,\nu} E_{m,\lambda}.$$

Morally speaking, this means that the expression (2.1) should belong to Eis_k , but the presence of nonholomorphic E_2 terms complicates the statement somewhat.

FAKE PROOF. We again ignore all issues of convergence, and work modulo “ Eis_k ” (ignoring the nonholomorphy coming from any E_2). Analogously to (1.4), we formally write the first term $\sum_{\ell+m=k} \alpha^{\ell-1} \beta^{m-1} E_{\ell,\lambda}(z) E_{m,\mu}(z)$ as the following sum over $((a, b), (c, d), (e, f)) \in T$ and $\ell, m \geq 1$ with $\ell + m = k$:

$$(2.2) \quad \sum_{((a,b),(c,d),(e,f))} \sum_{\ell, m} \frac{\alpha^{\ell-1} \beta^{m-1}}{(az+b)^\ell (cz+d)^m} = \sum_{((a,b),(c,d),(e,f))} \sum_{\ell, m} \frac{\alpha^{\ell-1} \beta^{m-1}}{p^\ell q^m},$$

using the notation of p, q, r as in (1.6) and the sentence that follows it. Some manipulation with the finite geometric series over ℓ, m then gives us the following congruence modulo “ Eis_k ”:

$$(2.3) \quad \sum_{\substack{\ell+m=k \\ \ell, m \geq 1}} \alpha^{\ell-1} \beta^{m-1} E_{\ell,\lambda} E_{m,\mu} \equiv \sum_{((a,b),(c,d),(e,f)) \in T} \frac{\left(\frac{\alpha}{p}\right)^{k-1} - \left(\frac{\beta}{q}\right)^{k-1}}{\alpha q - \beta p}.$$

Taking simultaneous cyclic permutations of (α, β, γ) and (p, q, r) , we obtain similar (purely formal) expressions for the second and third terms of (2.1). But now a miracle occurs: the identities $\alpha + \beta + \gamma = 0$ and $p + q + r = 0$ imply that

$$(2.4) \quad \alpha q - \beta p = \beta r - \gamma q = \gamma p - \alpha r.$$

(An amusing way to avoid verifying the above fact algebraically is to stick to real variables, and use the cross product in \mathbf{R}^3 : in that case, the vectors (α, β, γ) and (p, q, r) both lie in the plane orthogonal to $(1, 1, 1)$, so their cross product is parallel

to $(1, 1, 1)$, which gives precisely the equalities in (2.4).) Now adding up the cyclic permutations of the expressions on the right hand side of (2.3) gives a sum of the cyclic permutations of $\left(\frac{\alpha}{p}\right)^{k-1} - \left(\frac{\beta}{q}\right)^{k-1}$ over the **same** common denominator, and this immediately gives the desired sum of zero! \square

The identities between rational functions that we have used in the above two fake proofs, namely $1/pq + 1/qr + 1/rp = 0$ and the analogous cyclic sum involving also α, β, γ in higher weight, date back to Eisenstein; a good reference for this is Chapter II, section 2 and Chapter IV, section 1 of [Wei99] (but note that Weil uses $r = p + q$, whereas we use $r = -p - q$). The identities there, which are proved by a partial fraction decomposition and/or successive differentiation, may look more complicated than ours, particularly since they involve various sums and binomial coefficients. The relation to the identities we used above with auxiliary variables α, β, γ are however straightforward: write $\gamma = -\alpha - \beta$ in our identities, expand everything into a polynomial in α and β , then equate the coefficients of the same monomial $\alpha^\ell \beta^m$ on both sides to obtain the identities in [Wei99]. However, those identities, just like ours, always end up involving some terms with only a first power or square of p (or q , or r) in the denominator. This appears to make the convergence difficult to control, even if one sums both sides in Eisenstein style, with a sum $\sum_{(m,n) \in \mathbf{Z}^2}$ being carried out as $\lim_{M \rightarrow \infty} \sum_{m=-M}^M \lim_{N \rightarrow \infty} \sum_{n=-N}^N$. For that reason, the treatment in Chapter IV of [Wei99], following Eisenstein, proceeds by a somewhat different route.

3. An example related to Hecke operators

Another result of interest relates to the trace from $\Gamma(NM)$ to a lower level $\Gamma(M)$ of certain products of Eisenstein series. The basic expression for which we derive an identity in [KMR17] is a sum of the form $\sum_{\tau \in N^{-1}\mathbf{Z}^2/\mathbf{Z}^2} E_{\ell, \lambda + \tau} E_{m, \mu - S\tau}$, as explained in Section 4 of that article. Here $\lambda, \mu \in M^{-1}\mathbf{Z}^2/\mathbf{Z}^2$, and $S \in \mathbf{Z}$. The variable $\tau \in N^{-1}\mathbf{Z}^2/\mathbf{Z}^2$ in the sum can be thought of as a sum over all N -torsion points of the elliptic curve given analytically as $\mathbf{C}/(\mathbf{Z}z + \mathbf{Z})$. In this note, we will only deal with one example, but the result holds generally, as does the fake proof, in whatever sense fake proofs can be said to hold. The combinatorics are again reminiscent of the combinatorics one obtains when one computes Hecke operators on spaces of modular symbols, and involve sublattices of \mathbf{Z}^2 and the convex hull of the lattice points in the first quadrant; the references for this are Theorem 3.16 of [BG01] and its proof, Lemma 7.3 of [BG03], and Subsection 2.3 and Section 3 of [Mer94]. Here we will just illustrate these phenomena for the case $N = 5$ and $S = 3$, and make the connection with fake proofs based on interesting identities of rational functions.

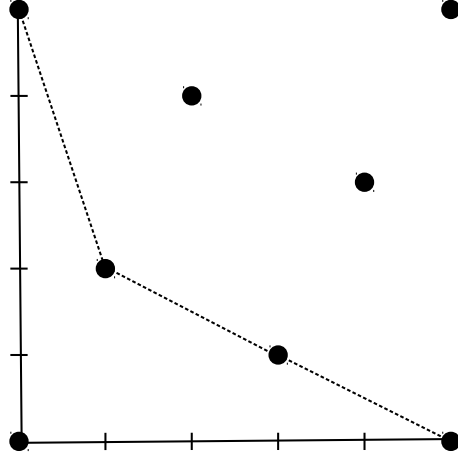
We thus fix $\lambda, \mu \in \mathbf{Q}^2$ (where usually only their image in $\mathbf{Q}^2/\mathbf{Z}^2$ matters). The exact level of λ, μ , i.e., their denominator, M is immaterial, since the formulas we obtain are insensitive to M . We then consider just the following identity, which is a special case of Proposition 4.1 of [KMR17] (and the notation $L_{\lambda, \mu, \alpha, \beta}$ is taken from there as well).

PROPOSITION 3.1. *Write $L_{\lambda, \mu, \alpha, \beta} = \sum_{\substack{\ell+m=k \\ \ell, m \geq 1}} \alpha^{\ell-1} \beta^{m-1} E_{\ell, \lambda} E_{m, \mu}$ for the expression in weight k that has appeared repeatedly in Section 2. Then, modulo “Eis_k”*

as usual, we have

$$(3.1) \quad \begin{aligned} & (1/5) \sum_{\tau \in 5^{-1}\mathbf{Z}^2/\mathbf{Z}^2} L_{\lambda+\tau, \mu-3\tau, \alpha, \beta} \\ & \equiv L_{5\lambda, 3\lambda+\mu, 5\alpha, 3\alpha+\beta} + L_{3\lambda+\mu, \lambda+2\mu, 3\alpha+\beta, \alpha+2\beta} + L_{\lambda+2\mu, 5\mu, \alpha+2\beta, 5\beta}. \end{aligned}$$

REMARK 3.2. The vectors $(5, 0)$, $(3, 1)$, $(1, 2)$, and $(0, 5)$ in sequence are obtained by taking the convex hull of the nonzero points in the first quadrant of the sublattice $\{(x, y) \mid x - 3y \equiv 0 \pmod{5}\}$ of \mathbf{Z}^2 , as in the figure below. Any pair of consecutive vectors has a determinant of 5, the index. This is described further in the references mentioned above.



FAKE PROOF OF PROPOSITION 3.1. It is clearer if we first restrict to $k = 2$, in which case everything related to α, β can be omitted, and $L_{\lambda, \mu, \text{anything}} = E_{1, \lambda} E_{1, \mu}$. We formally expand $\sum_{\tau} E_{1, \lambda+\tau} E_{1, \mu-3\tau}$ using as usual $v = (a, b) \equiv \lambda + \tau$ and $w = (c, d) \equiv \mu - 3\tau \pmod{\mathbf{Z}^2}$, taking into account also the sum over τ . Write as usual $p = az + b$ and $q = cz + d$; these depend linearly on v and w . Our left hand side is thus the sum of all terms $1/(5pq)$, as $(v, w) = (\lambda + v', \mu + w') \in \mathbf{Q}^4$ ranges over all possible shifts of (λ, μ) by the lattice $\Lambda = \{(v', w') \in 5^{-1}\mathbf{Z}^4 \mid 3v' + w' \in \mathbf{Z}^4\}$.

For each two consecutive vectors in the list $\{(5, 0), (3, 1), (1, 2), (0, 5)\}$, say for example the vectors $(3, 1)$ and $(1, 2)$, one can see that as (v', w') varies over Λ , the resulting combination $(3v' + w', v' + 2w')$ (made using the coefficients of the two consecutive vectors) varies precisely over all of \mathbf{Z}^4 . It follows that the pair of values $(3p + q, p + 2q)$ varies over the terms in such a way that the (nonconvergent, as always) sum of all the products $(3p + q)^{-1}(p + 2q)^{-1}$ yields $E_{1, 3\lambda+\mu} E_{1, \lambda+2\mu}$. It now remains to make use of the hopefully impressive identity

$$(3.2) \quad \frac{1}{5pq} = \frac{1}{(5p)(3p+q)} + \frac{1}{(3p+q)(p+2q)} + \frac{1}{(p+2q)(5q)}$$

to conclude the fake proof for $k = 2$. In all this, we have blithely ignored the fact that in all our sums, we omitted any terms that look like $1/0$, which may have introduced correction terms that with luck will belong to Eis_2 ; as mentioned at the end of Section 1, however, the issues with convergence seem to produce further unavoidable corrections from Eis_2 , even if the above formal argument has not omitted any terms (e.g., if all of $5\lambda, 3\lambda + \mu, \lambda + 2\mu, 5\mu$ are nonzero in $\mathbf{Q}^2/\mathbf{Z}^2$).

We note that the identity (3.2) may become more apparent if we observe that

$$\begin{aligned}
 (3.3) \quad & \frac{1}{(5p)(3p+q)} = \frac{1/q}{5p} - \frac{3/(5q)}{3p+q}, \\
 & \frac{1}{(3p+q)(p+2q)} = \frac{3/(5q)}{3p+q} - \frac{1/(5q)}{p+2q}, \\
 & \frac{1}{(p+2q)(5q)} = \frac{1/(5q)}{p+2q} - \frac{0}{5q}, \\
 & \text{generally, } \frac{1}{(ap+bq)(cp+dq)} = \frac{a/((ad-bc)q)}{(ap+bq)} - \frac{c/((ad-bc)q)}{(cp+dq)}.
 \end{aligned}$$

Thus when we add the terms coming from each consecutive pair vectors in the list $\{(N, 0), \dots, (0, N)\}$, all the middle terms cancel, and we are left with $1/(Npq)$.

We now turn to the fake proof for arbitrary weight k . For general k , the sum giving the left hand side of (3.1), $(1/5) \sum_{\tau} L_{\lambda+\tau, \mu-3\tau, \alpha, \beta}$, is a sum not merely of $1/(5pq)$, but rather, as in (2.3), of $[(\alpha/p)^{k-1} - (\beta/q)^{k-1}]/[5(\alpha q - \beta p)]$. The sum runs over the same collection of p, q as before, and the same type of combinations (using consecutive vectors in the list $\{(5, 0), \dots\}$) relate the lattice Λ to \mathbf{Z}^4 , specifically for the terms appearing on the right hand side of (3.1). One obtains that the three L expressions there amount to summing each of the following three terms:

$$\begin{aligned}
 (3.4) \quad & \frac{(5\alpha/5p)^{k-1} - ((3\alpha + \beta)/(3p + q))^{k-1}}{(5\alpha)(3p + q) - (3\alpha + \beta)(5p)}, \\
 & \frac{((3\alpha + \beta)/(3p + q))^{k-1} - ((\alpha + 2\beta)/(p + 2q))^{k-1}}{(3\alpha + \beta)(p + 2q) - (\alpha + 2\beta)(3p + q)}, \\
 & \frac{((\alpha + 2\beta)/(p + 2q))^{k-1} - (5\beta/5q)^{k-1}}{(\alpha + 2\beta)(5q) - (5\beta)(p + 2q)}.
 \end{aligned}$$

Once again, a minor miracle occurs in that the denominators of the terms are all equal to the same expression, namely $5(\alpha q - \beta p)$, so we formally get the desired left hand side. The same phenomenon happens in general, not just for $N = 5$. \square

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